Scattering of Yang-Mills Quanta*

EDWARD A. REMLER
Institute of Field Physics, University of North Carolina, Chapel Hill, North Carolina
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Dirac spinor quanta interacting via a Yang-Mills field are considered and lowest order perturbation theoretic transition matrix elements are given for the elementary scattering processes. All expressions are invariant under the full gauge group. All processes exhibit infinite forward scattering as expected from the masslessness and charge of the Yang-Mills field quanta. A helicity conservation law holds when Yang-Mills quanta scatter each other.

I. INTRODUCTION

THE Yang-Mills (YM) field[1] is a generalization of the electromagnetic field, associated with a corresponding generalization of the gauge group of electrodynamics, which has become of interest to people working in elementary particle theory[2] and also to those examining problems in connection with generally relativistic quantum field theories.[3] This paper gives the lowest order perturbation theoretic transition matrix elements implied by the YM generalization of the electrodynamic interaction.

The best way to deal with systems possessing these more complicated gauge groups is not yet clear.[4] It is not necessary, however, to develop a complete theory of gauge invariant quantum fields in order to calculate what may reasonably be expected in lowest order. Thus, if one writes down the full gauge invariant action and then simply uses the interaction part of it[5] as the generator of the scattering matrix in the interaction representation, it can be shown[6] in general that the sum of all diagrams, having *no closed loops* in each order of perturbation theory, is gauge invariant.[7] As a particular example of the general validity of this statement, one finds that the two expressions calculated here are gauge invariant.

In Sec. II we introduce the basic expressions and notation to be used; Sec. III deals with the YM generalization of Compton scattering; Sec. IV deals with the scattering of one YM quantum by another via a mechanism absent in the electrodynamic case; Sec. V deals with the YM generalization of Moller scattering.

II. BASIC EXPRESSIONS AND NOTATION

The fields with which we are concerned are a *set* of spinor fields (spinor indices are always suppressed) ψ_a, $a=1\cdots N$, interacting with a *set* of four-vector fields, $A_{r\mu}$, $r=1\cdots n$, $\mu=0, 1, 2, 3$, which are the YM generalizations of the electromagnetic potentials. These fields are to be associated with some particular simple Lie group[8] and one of its unitary representations. The indices attached to the YM field $A_{r\mu}$ range over the dimensionality (n) of the underlying Lie group. The indices attached to the Dirac field range over the degree (N) of the particular representation considered. Both types of indices will be suppressed in the following wherever possible.

Because of our restriction to simple Lie groups we can, without any loss of physical generality, always use a coordinatization of the Lie group such that the structure constants obey

$$c_{rst}=c_{trs}=-c_{srt}\,,\quad \text{etc.}\,, \tag{1}$$

$$c_{rst}\equiv(c_s)_{rt}\,, \tag{2}$$

$$\mathrm{tr}\,c_r c_{r'}=-e^2\delta_{rr'}\,, \tag{3}$$

$$[c_r,c_s]=c_{rst}c_t\,. \tag{4}$$

The unitary representation is determined by its anti-Hermitian generators $(M_r)_{ab}$, the generalizations of the quantity (ie) of electrodynamics, obeying the group law

$$[M_r,M_s]=c_{rst}M_t\,. \tag{5}$$

The vector cross product, associated with the three-dimensional rotation group may be generalized in the following manner:

$$A\times B\rightarrow AcB\,,$$
$$(AcB)_r\equiv A_sB_tc_{rst}=-(BcA)_r\,, \tag{6}$$

$$AB\equiv A_rB_r\,, \tag{7}$$

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[1] First introduced by C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954). See also R. Utiyama, Phys. Rev. **101**, 1597 (1956) and S. Glashow and M. Gell-Mann, Ann. Phys. (N. Y.) **15**, 437 (1961) for further generalizations and more details on the rationale being the introduction of this field.

[2] S. Glashow and M. Gell-Mann, Ref. 1; J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960).

[3] B. S. Dewitt, J. Math. Phys. **3**, 1077 (1962).

[4] B. S. DeWitt, in Relativity, Groups and Topology, Les Houches Summer School Lectures (to be published). J. Schwinger, Phys. Rev. **125**, 1043 (1962).

[5] Terms cubic and quadratic in the fields in this case.

[6] This was first shown by Feynman and is usually called the "tree theorem." A proof is given by DeWitt, Ref. 4.

[7] By an expression being "gauge invariant" we shall mean one which depends on the functions describing single-particle states of YM quanta in a physical process and vanishes when for any of these we use a function of the form $f_\alpha(x)=\partial_\alpha f(x)$. Note. We do *not* restrict $f(x)$ by a subsidiary condition. Thus, we speak of invariance under the "full" gauge group and not under one of its subgroups.

[8] The terminology and theorems used in his paper with respect to group theory may be found in G. Racah, "Group Theory and Spectroscopy" (unpublished lecture notes), Institute for Advanced Study.

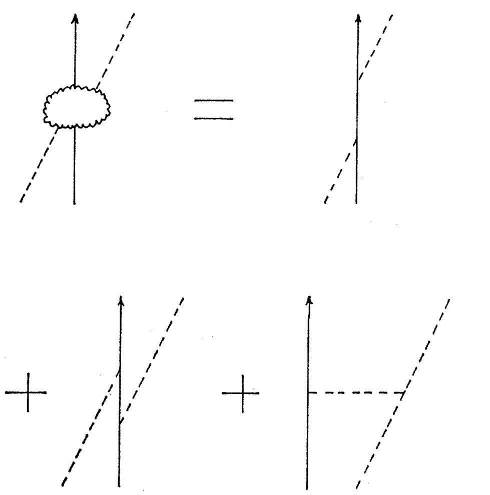

FIG. 1. Types of diagrams contributing to generalized Compton scattering. Solid line indicates Fermions; dashed line indicates YM quanta.

$$ABcC=A_r(BcC)_r\,, \tag{8}$$

$$ABcC=AcBC=CAcB\,, \tag{9}$$

$$Ac(BcC)+Cc(AcB)+Bc(CcA)=0\,. \tag{10}$$

In terms of these conventions, the action under consideration[9] may be written as

$$S=-\int\bar{\psi}[\gamma(\partial+AM)+m]\psi(dx) -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}(dx)\,, \tag{11}$$

$$F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu+A_\mu cA_\nu\,, \tag{12}$$

$$\gamma\partial=\eta^{\alpha\beta}\gamma_\alpha\partial_\beta\,,$$

$$\eta^{\alpha\beta}\equiv\operatorname{diag}(-1,\,1,\,1,\,1)\,.$$

This action is invariant under the YM gauge transformation which can be defined by its infinitesimal law,

$$\delta\psi=\Lambda_r M_r\psi=\Lambda M\psi\,, \tag{13}$$

$$\delta A_\mu=-\partial_\mu\Lambda+\Lambda cA_\mu\,, \tag{14}$$

$$\delta F=\Lambda cF\,, \tag{15}$$

$$\Lambda_r=\Lambda_r(x)\,.$$

The action (11) cannot be used directly as the basis of a canonical quantization procedure because of the very gauge invariance for which it was created. We can, for the limited purposes of this paper, eliminate this formal embarrassment by imitating the procedure used in electrodynamics and striking out the term $+\frac{1}{2}\int(\partial_\mu A_\nu\partial^\nu A^\mu)(dx)$ in (11). The resulting action admits canonical quantization. The Hamiltonian in the interaction representation consists of minus the interaction part of the Lagrangian in (11) plus a "normal dependent" part the latter of which may be simply ignored if one uses the method of P^* symbols.[10] We obtain an effective interaction representation Hamiltonian then

$$H^{\text{eff}}=\int(dx^1dx^2dx^3)\{\bar{\psi}\gamma AM\psi+\partial_\mu A_\nu(A^\mu cA^\nu) +\tfrac{1}{4}(A_\mu cA_\nu)(A^\mu cA^\nu)\}\,. \tag{16}$$

The fields obey,

$$[A_\mu,A_\nu']=-i\eta_{\mu\nu}D(xx')\,, \tag{17}$$

$$\{\psi,\bar{\psi}'\}=iS(xx')\,. \tag{18}$$

To order e^2 the scattering operator is

$$P^*\left\{\exp-i\int_{-\infty}^{+\infty}H^{\text{eff}}dt\right\} \approx P^*\left[-(i/4)\int A_\mu cA_\nu A^\mu cA^\nu(dx)\right] +P^*\left[-\frac{1}{2}\int(\bar{\psi}\gamma AM\psi+\partial_\mu A_\nu A^\mu cA^\nu) \times(\bar{\psi}'\gamma A'M\psi'+\partial_\mu'A_\nu'A'^\mu cA'^\nu)(dx)(dx')\right]. \tag{19}$$

There are three essentially different classes of scattering processes predicted by (19), namely, those involving none, one, or two real Dirac quanta. Since the details of these calculations involve nothing new, they will be omitted.[11] The results apply to the case when the YM field transforms according to any simple Lie group

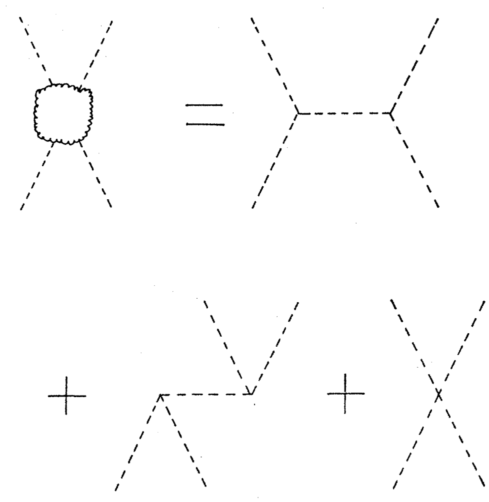

FIG. 2. Types of diagrams contributing to the scattering of one YM quanta by another.

[9] We omit repetition of the derivation by induction of this action which is fully covered in the references cited in Refs. 1 and 4.

[10] K. Nishijima, Progr. Theoret. Phys. (Kyoto) 5, 405 (1950).

[11] These are outlined in E. Remler, Doctoral dissertation, University of North Carolina, 1963 (unpublished).

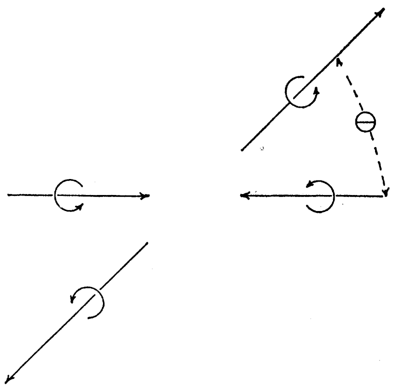

FIG. 3. Diagram showing case in which both YM quanta have positive helicity. Invariance under space reflections gives the same results for negative helicity. Incoming and outgoing states have the same helicity.

while the Dirac field, interacting with it, transforms according to any unitary representation of this Lie group.

III. GENERALIZED COMPTON SCATTERING

The Feynman diagrams contributing to this process, in which a YM quantum scatters with a Dirac quantum, are shown in Fig. 1. The last diagram is seen to be the essentially new feature of the general case. It has a vertex expressing the self-interacting qualities of the YM field which only vanishes in the degenerate case of electrodynamics. These diagrams give the following expression for the transition matrix:

$$\begin{aligned}\langle p'k'|T|pk\rangle = i\int (dx)(dx')\{&D_c(xx')[D'(x')\gamma_\alpha MD(x')]\\ &\times[(2\partial_\beta Y_\alpha(x)cY_\beta'(x)-\partial_\alpha Y_\beta(x)cY_\beta'(x))+(Y\leftrightarrow Y')]\\ &-D'(x)[(\gamma Y(x)M)S_c(xx')(\gamma Y'(x')M)\\ &\qquad+Y(x)\leftrightarrow Y'(x')]D(x)\}. \quad (20)\end{aligned}$$

Y and Y' describe the initial and final YM quanta eigenstates; D and D' describe the initial and final Dirac quanta eigenstates. These are of the form

$$\begin{aligned}Y_{r\mu}(x)&=(2\pi)^{-3/2}(2\omega)^{-1/2}e^{ikx}y_r\sigma_\mu,\\ Y_{r\mu}'(x)&=(2\pi)^{-3/2}(2\omega')^{-1/2}e^{-ik'x}y_r'\sigma_\mu',\\ D_a(x)&=(2\pi)^{-3/2}(m/\epsilon)^{1/2}e^{ipx}z_a u,\\ D_a'(x)&=(2\pi)^{-3/2}(m/\epsilon')^{1/2}e^{-ip'x}z_a'u'.\end{aligned}$$

In these formulas the u are spinors, the σ are polarization 4-vectors, the y are vectors in Lie group space, the z are vectors in a representation space of the Lie group, the ω and ω' are the initial and final YM quanta energies, the ϵ and ϵ' are the initial and final Dirac quanta energies, S_c and D_c are causal propagators,[12]

[12] Defined in J. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955).

and

$$\begin{aligned}p^2&=-m^2,\\ k^2&=0,\\ (px)&\equiv p_\mu x^\mu.\end{aligned}$$

A short calculation shows that the right-hand side of (20) is gauge invariant.

The transition matrix element for laboratory system scattering implied by Eq. (20) is

$$\begin{aligned}\langle p'k'|T|pk\rangle=&[4\pi(\epsilon\epsilon')^{1/4}(\omega\omega')^{3/4}]^{-2}\delta^4(p+k-p'-k')\\ &\times[M_S+M_A(\omega'+\omega)/(\omega'-\omega)]\\ &\times\bar{u}'[\omega'(\gamma\sigma')(\gamma\sigma)(\gamma k)+\omega(\gamma\sigma)(\gamma\sigma')(\gamma k')]u, \quad (21)\end{aligned}$$

where

$$\begin{aligned}M_S&\equiv\tfrac{1}{2}z^{*\prime}[(y^{*\prime}M)(yM)+(yM)(y^{*\prime}M)]z,\\ M_A&\equiv\tfrac{1}{2}z^{*\prime}[(y^{*\prime}M)(yM)-(yM)(y^{*\prime}M)]z,\\ &=\tfrac{1}{2}(z^{*\prime}Mz)(y^{*\prime}cy). \quad (22)\end{aligned}$$

The last line of (22) comes directly from (5). The scattering cross section in the laboratory system, averaged over initial and summed over final Dirac quantum spin states, can easily be obtained from the corresponding formula in electrodynamics by making the substitution

$$e^2\to|M_S+M_A(\omega'+\omega)/(\omega'-\omega)|^2. \quad (23)$$

It is not hard to show directly, using (4) and (5), that M_A and M_S are invariant under the subgroup of the full gauge group generated by parameters of the form $\Lambda(x)=$constant. This implies conservation of "inner space" quantum numbers such as hypercharge and isospin. The term proportional to $|M_S|^2$ in the formula for the cross section is just the Klein-Nishina formula, the remaining terms, containing the structure constants and giving infinite forward scattering of the Coulomb type, stem from the noncommutativity of the generators and are absent only in the degenerate case of electrodynamics, the gauge group of which is the only simple abelian Lie group.

IV. SELF-SCATTERING OF YM QUANTA

The Feynman diagrams contributing to this process, in which a YM quantum scatters with another YM quantum, are shown in Fig. 2. The scattering matrix element can be written as

$$\begin{aligned}\langle k''''k'''|T|k''k'\rangle\\ =\mathrm{Perm}\int(dx)(dx')iD_c(xx')[2(\partial_\beta Y_\alpha'(x)\\ -\partial_\alpha Y_\beta'(x))cY_\beta''(x)\partial_\gamma Y_\alpha'''(x')cY_\gamma''''(x')\\ +\tfrac{1}{2}\partial_\alpha Y_\beta'(x)cY_\beta''(x)\partial_\alpha Y_\gamma'''(x')cY_\gamma''''(x')]\\ +\mathrm{Perm}\int(dx)(-i/4)Y_\alpha'(x)\\ \times cY_\beta''(x)Y_\alpha'''(x)cY_\beta''''(x). \quad (24)\end{aligned}$$

In (24), Perm denotes summation over all permutations of the Y (the space time indices x, x' as well as the suppressed discrete indices of the Y functions are of course not permuted). A slightly more lengthy calculation than in the Compton scattering case shows that (24) is gauge invariant.

Equation (24) has been calculated for center-of-mass system scattering of transversely polarized incoming and outgoing YM quanta, i.e., with

$$\begin{aligned} k_\alpha' &= (\omega,k_1,k_2,k_3)=(\omega,\mathbf{k}),\\ k_\alpha'' &= (\omega,-\mathbf{k}),\\ k_\alpha''' &= -(\omega,\mathbf{k}'),\\ k_\alpha'''' &= -(\omega,-\mathbf{k}'),\\ \boldsymbol{\sigma}'\cdot\mathbf{k} &= \boldsymbol{\sigma}''\cdot\mathbf{k}=\boldsymbol{\sigma}'''\cdot\mathbf{k}'=\boldsymbol{\sigma}''''\cdot\mathbf{k}'=0. \end{aligned}$$

There are only two essentially different spin states of the two quanta ingoing or outgoing system. These are shown in Figs. 3 and 4. In Fig. 3 both quanta have the same helicity; in Fig. 4 each have different helicity. All other situations can be obtained from these two by space reflections and rotations. It is a salient feature of the result that the number of quanta of each helicity is conserved in scattering. There are then only two different types of transition matrix elements, namely,

$$\langle 34|T|12\rangle = i(4\pi\omega)^{-2}\delta^4(k'+k''+k'''+k'''')\times(c_A\cos\theta-c_S)\left\{\begin{matrix}(2/\sin\theta)^2\\ ((1+\cos\theta)/\sin\theta)^2\end{matrix}\right\}. \quad (25)$$

In (25) the upper possibility refers to the case in which the initial and final quanta all have positive or all have negative helicity—the one illustrated in Fig. 3. The lower possibility refers to the case in which each incoming and each outgoing particle differ in helicity—the one illustrated in Fig. 4. The definition of θ is given in the figures. Finally, information about the states of the scattering particles, with respect to their inner space indices, is contained in the expressions

$$\begin{aligned} c_S &\equiv (y'cy''')(y''''cy'')+(y'cy'''')(y'''cy''),\\ c_A &\equiv (y'cy''')(y''''cy'')-(y'cy'''')(y'''cy''),\\ &= (y'cy'')(y'''cy''''). \end{aligned}$$

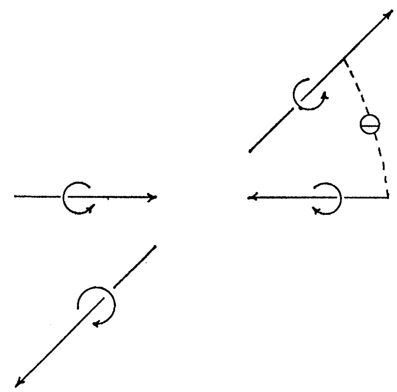

FIG. 4. Diagram showing case in which each YM quanta has a different helicity. The angle θ is defined as the angle between the incoming and outgoing quanta having the same helicity.

V. GENERALIZED MOLLER SCATTERING

The Feynman diagrams contributing to this process, in which a Dirac particle scatters with another such, are of the same form as in electrodynamics. At each vertex one now has $(M_r\gamma_\mu)$ instead of $(ie\gamma_\mu)$ as in electrodynamics. When finding the cross section summed and averaged over spin states, the trace computations are exactly those of electrodynamics. Consequently, one can take the notation and results directly from those given in Ref. 12, Chap. 12 on Moller scattering. Then [see Eq. (12-1), Ref. 12],

$$\begin{aligned} \langle p_1'p_2'|T|p_1p_2\rangle &= \frac{im^2}{(2\pi)^2}(\epsilon_1\epsilon_1'\epsilon_2\epsilon_2')^{-1/2}\delta^4(p_1+p_2-p_1'-p_2')\\ &\times\{(M_S+M_A)(p_1-p_1')^{-2}(\bar{u}_1'\gamma u_1)(\bar{u}_2'\gamma u_2)\\ &-(M_S-M_A)(p_1-p_2')^{-2}(\bar{u}_2'\gamma u_1)(\bar{u}_1'\gamma u_2)\} \quad (26) \end{aligned}$$

while now, in analogy with our previous notation, one can write

$$\begin{aligned} M_S &\equiv \tfrac{1}{2}[(z_1^{*\prime}M_rz_1)(z_2^{*\prime}M_rz_2)+(z_2^{*\prime}M_rz_1)(z_1^{*\prime}Mz_2)],\\ M_A &\equiv \tfrac{1}{2}[(z_1^{*\prime}M_rz_2)(z_2^{*\prime}M_rz_1)-(z_2^{*\prime}M_rz_1)(z_1^{*\prime}M_rz_2)]. \end{aligned}$$

The part proportional to M_S gives then essentially the Moller scattering formula. The generalized Bhabha scattering formula is equally obvious.